

Reply to 'Comment on "Dynamic correlations of the spinless Coulomb Luttinger liquid [Phys. Rev. B 65, 125109 (2002)]"'

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We show that the criticism of our paper [Phys. Rev. B 65, 125109 (2002)] by Wang, Millis, and Das Sarma [cond-mat/0206203] is based on a trivial mathematical mistake they have committed.

Coulomb interaction in one-dimensional electron systems arouses firm and enduring interest for two reasons. First, this is a strong interaction of clear and fundamental nature. Second, the Coulomb interaction case is usually difficult to treat mathematically, and therefore exact solutions are always important.

We have investigated the Luttinger liquid (LL) with Coulomb electron-electron interaction in our paper [1]. The main result of our work is apparently the existence of a soft collective charge mode, related to dynamic electron correlations of the $2k_F$ scale. Ref. [1] presents also an analytic method to study various dynamic correlation functions near the threshold.

Wang, Millis, and Das Sarma (WMS henceforth) claim in their Comment [2] that our method, as well as ensuing results, is incorrect. The claim is based on WMS's abortive attempt to reproduce our calculation of the CDW structure factor. WMS substitute our expression for the structure factor into the integral equation, which we derived for it, and conclude that corrections to our result are given by diverging integrals. After that, WMS accuse our method to be 'inconsistent with mathematical analysis'.

In this Reply, we show that WMS have committed a trivial mathematical mistake in their calculations. Diverging integrals appear in Ref. [2] just because WMS have incorrectly differentiated the structure factor.

Recall that the CDW structure factor $S(q, \omega)$ contains the Heaviside function $\theta(\omega - \omega_q)$ as a factor, which reflects the existence of the threshold.³ We stress that $\theta(\omega - \omega_q)$ depends on ω and q , and hence must be differentiated when finding the derivative of $S(q, \omega)$. WMS disregard this fact and differentiate $S(q, \omega)$, ignoring θ -function. Below we demonstrate that if one acts correctly, then all the integrals of our paper [1] are well-defined, and expansions are convergent.

In Ref. [1] we have shown that $S(q, \omega)$ satisfies the following integral equation:

$$\frac{\omega}{v_F} S(q, \omega) = \int_{-\infty}^{+\infty} dQ S(Q, \omega - \omega_{q+Q}). \quad (1)$$

Here ω_q is the energy of the LL bosons, $\omega_q = v_F |q|/g(q)$, v_F is the Fermi velocity, $g(q)$ is the interaction parameter.⁴ For Coulomb interaction $g(q) = \beta |\ln |q|d|^{-1/2}$, with d being the diameter of a quantum wire, $\beta = [\pi \hbar v_F / 2e^2]^{1/2}$. For simplicity, the wave number q is measured from $2k_F$.

Expand $S(Q, \omega - \omega_{q+Q})$ in the powers of the wave number Q , appearing in the second (frequency) argument, to

get

$$\frac{\omega}{v_F} S(q, \omega) = \int_{-\infty}^{+\infty} dQ [S(Q, \omega - \omega_q) + \frac{Q^2}{2!} S_{qq}(Q, \omega - \omega_q) + \dots]. \quad (2)$$

Restricting expansion (2) to the first term on the RHS, we have found in Ref. [1] that

$$S(q, \omega) = \frac{v_F}{\omega} \frac{e^{-4\beta |\ln \epsilon|^{1/2}}}{\epsilon |\ln \epsilon|^{1/2}} \theta(\epsilon), \quad (3)$$

where we denoted $\epsilon = \omega - \omega_q$. The structure factor $S(q, \omega)$ is seen to be zero for $\epsilon < 0$, and to diverge as $\epsilon \rightarrow +0$.

In our paper [1] we have emphasized that the Coulomb interaction case is special in the respect that retaining only the first term on the RHS of Eq. (2) already gives the correct asymptotic behavior of $S(q, \omega)$ at $\epsilon \rightarrow 0$.

On the contrary, WMS find in Ref. [2] that the second term on the RHS of Eq. (2) is infinite, and conclude that the expansion (2) does not exist at all.

Let us prove that the second term of the expansion (2) is well-defined, and corrections to our result (3) are indeed small. Substitute $S(q, \omega)$ from Eq. (3) into Eq. (2), as WMS wish [2], but do it correctly at this time. The second term of the expansion becomes

$$\begin{aligned} & \partial_{qq} \int_{-\infty}^{+\infty} dQ \frac{Q^2}{2!} S(Q, \epsilon) \\ &= \partial_{qq} \int_0^{+\infty} dQ Q^2 \frac{v_F}{\epsilon} \left[\frac{e^{-4\beta |\ln \delta|^{1/2}}}{\delta |\ln \delta|^{1/2}} \right] \theta(\delta), \end{aligned} \quad (4)$$

where we denoted $\delta = \epsilon - \omega_q$. It is important that both ϵ and δ depend on q , which must be taken into account when calculating the derivative ∂_{qq} . When $\epsilon < 0$, δ is always negative, and expression (4) is zero due to the factor $\theta(\delta)$. If $\epsilon > 0$, then δ varies from ϵ to 0. Take δ as a new integration variable. Eq. (4) can be written as

$$\partial_{qq} \int_0^\epsilon d\delta \frac{\beta^3}{v_F^2} \frac{(\epsilon - \delta)^2}{\epsilon |\ln(\epsilon - \delta)|^{3/2}} \left[\frac{e^{-4\beta |\ln \delta|^{1/2}}}{\delta |\ln \delta|^{1/2}} \right] \theta(\delta). \quad (5)$$

As WMS note [2], the most important contribution to the integral comes from the region $\delta \sim 0$, where the expression in the square brackets of Eq. (5) diverges. For this reason we can replace all $(\epsilon - \delta)$ in the integral with ϵ .

Since the expression in the square brackets equals identically $(2\beta)^{-1}\partial_\delta(\exp(-4\beta|\ln\delta|^{1/2}))$, the total formula (5) is

$$\partial_{qq} \left[\frac{\beta^2}{2v_F^2} \epsilon \frac{e^{-4\beta|\ln\epsilon|^{1/2}}}{|\ln\epsilon|^{3/2}} \theta(\epsilon) \right]. \quad (6)$$

We underline that this expression is a well-defined distribution, rather than a classic function. Distributions normally arise when expanding a function with a threshold into Taylor's series. Hence it is not surprising that the higher order terms of the expansion diverge. Simply they are not 'corrections' as WMS suppose. In calculating corrections to our result, one cannot retain only two terms of the expansion (2). The correct procedure [5] requires summing up the total series in Eq. (2) to get the result in terms of classic functions. We have performed such calculation, and have not included the result into our paper [1], since it consists in a not too important replacement of the argument in the expression (3) for $S(q, \omega)$. The argument $\epsilon = \omega - \omega_q$ should be shifted by $\omega'_q \epsilon / |\ln\epsilon|^{1/2}$, which shift is obviously negligible as $\epsilon \rightarrow 0$.

Thus we confirm our result that Eq. (3) gives the asymptotic behavior of $S(q, \omega)$ as $\epsilon \rightarrow +0$. WMS's

claim [2] that the true diverging behavior should not be like Eq. (3) is misleading and incorrect.

Now consider the *mistake*, which is the basis of the WMS Comment [2]. In estimating the integral (4), WMS first calculate the derivative of the integrand w.r.t. q . This way of calculation, though less economical than the one presented above, would nevertheless lead them to correct result, provided that WMS calculate the derivative correctly. They do not take into account that the argument $\delta = \omega - \omega_q - \omega_Q$ of the $\theta(\delta)$ -function depends on q , using implicitly the incorrect relation

$$\frac{\partial^2}{\partial^2 q} \left[\frac{e^{-4\beta|\ln\delta|^{1/2}}}{\epsilon\delta|\ln\delta|^{1/2}} \theta(\delta) \right] = \theta(\delta) \frac{\partial^2}{\partial^2 q} \left[\frac{e^{-4\beta|\ln\delta|^{1/2}}}{\epsilon\delta|\ln\delta|^{1/2}} \right],$$

which finally leads them to conclusion that the integral (4) diverges (see the showy, yet erroneous Eq. (14) of Ref. [2]). In regards to WMS's appeal for the purity of the mathematical analysis, there is nothing left but to refer them to the important work of Leibniz [6], which explains how the derivative of the function product should be found.

¹ Yasha Gindikin and V.A. Sablikov, Phys. Rev. B **65**, 125109 (2002).

² D.W. Wang, A.J. Millis, and S. Das Sarma, cond-mat/0206203 (unpublished).

³ A. Luther and I. Peschel, Phys. Rev. B **9**, 2911 (1974).

⁴ J. Voit, Rep. Prog. Phys., **58**, 977 (1995).

⁵ I.M. Gelfand and G.E. Shilov, *Distributions and actions over them*, (Moscow, 1959).

⁶ G.W. Leibniz, Acta Eruditorum, (1684).